

Prevalence problem in the set of quadratic stochastic operators acting on L^1

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Abstract

This paper is devoted to the study of the problem of prevalence in the class of quadratic stochastic operators acting on the L^1 space for the uniform topology. We obtain that the set of norm quasi-mixing quadratic stochastic operators is a dense and open set in the topology induced by a very natural metric. This shows the typical long-term behaviour of iterates of quadratic stochastic operators.

Keywords : Quadratic stochastic operators, Nonhomogeneous Markov operators, Baire category, Mixing nonlinear Markov process

1 Introduction

Iterates of Markov operators have been studied for a long time due to their wide range of applications in different areas of science and technology. Today, motivated by numerous biological and physical phenomena, there is a growing interest in nonlinear methods. Recently Kolokoltsov (2010) wrote a detailed overview of the theory of nonlinear Markov processes. Here we study the so-called quadratic stochastic operators which are bilinear by definition. They were first introduced by Bernstein (1924) to describe the evolution of a discrete probability distribution of a finite number of biotypes in a process of inheritance. Since then the field has been steadily evolving and Kesten (1970); Lyubich (1992) provide a good overview of it. Furthermore Ganikhodzhaev et al. (2011) discuss a number of open problems in it.

A typical question when working with quadratic stochastic operators is the long-term behaviour of their iterates. Bartoszek and Pułka (2013b) introduced and studied different types of asymptotic behaviours of quadratic stochastic operators acting on the ℓ^1 space. These results were subsequently extended to the L^1 case by us (Bartoszek and Pułka, 2015). In this paper we are especially interested in answering the question what a typical (generic with respect to a specified metric topology) quadratic stochastic operator acting on L^1 looks like, i.e. we search for prevalent subsets in the class of quadratic stochastic operators. However, our work is more than a continuation of our previous results (Bartoszek and Pułka, 2015). In our previous paper (Bartoszek and Pułka, 2015) we showed equivalent conditions for asymptotic stability of quadratic stochastic operators of a very special type, namely kernel ones. Here we do not restrict ourselves to this class but study the geometry of the set of all quadratic stochastic operators.

2 Basic concepts

In this section we introduce notation alongside some basic definitions and properties. Let (X, \mathcal{A}, μ) be a separable σ -finite measure space. Throughout the paper by L^1 we denote the (separable) Banach lattice of real and \mathcal{A} -measurable functions f such that $|f|$ is μ -integrable and equipped with the norm $\|f\|_1 := \int_X |f| d\mu$. By $\mathcal{D} := \mathcal{D}(X, \mathcal{A}, \mu)$ we denote the convex set of all *densities* on X , i.e.

$$\mathcal{D} = \{f \in L^1 : f \geq 0, \|f\|_1 = 1\}.$$

We say that a linear operator $P: L^1 \rightarrow L^1$ is *Markov* (or *stochastic*) if

$$Pf \geq 0 \quad \text{and} \quad \|Pf\|_1 = \|f\|_1$$

for all $f \geq 0$, $f \in L^1$. Clearly $\|P\| := \sup_{\|f\|_1=1} \|Pf\|_1 = 1$ and $P(\mathcal{D}) \subset \mathcal{D}$. The sequence of such operators denoted by $\mathbf{P} = (P^{[n, n+1]})_{n \geq 0}$ is called a (*discrete time*) *nonhomogeneous chain of stochastic operators* or shorter, a *nonhomogeneous Markov chain*. For $m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n - m \geq 1$, and any $f \in L^1$ we set

$$P^{[m, n]} f = P^{[n-1, n]}(P^{[n-2, n-1]}(\dots(P^{[m, m+1]} f) \dots)).$$

We denote by I the identity operator and naturally $P^{[n, n]} = I$. If for all $n \geq 0$ one has $P^{[n, n+1]} = P$, then we say that $\mathbf{P} = (P)_{n \geq 0}$ is *homogeneous*. The set of all chains of Markov operators $\mathbf{P} = (P^{[n, n+1]})_{n \geq 0}$ will be denoted by \mathfrak{S} .

Different types of asymptotic behaviours as well as residualities in the set \mathfrak{S} (endowed with suitable natural metric topology) have been recently intensively studied by Pułka (2011, 2012). Following Pułka (2012) we introduce the below types of asymptotic stabilities.

Definition 2.1 *A discrete time nonhomogeneous chain of stochastic operators $\mathbf{P} \in \mathfrak{S}$ is called*

1. *uniformly asymptotically stable if there exists a unique $f_* \in \mathcal{D}$ such that for any $m \geq 0$*

$$\lim_{n \rightarrow \infty} \sup_{f \in \mathcal{D}} \left\| P^{[m,n]} f - f_* \right\|_1 = 0,$$

2. *almost uniformly asymptotically stable if for any $m \geq 0$*

$$\lim_{n \rightarrow \infty} \sup_{f, g \in \mathcal{D}} \left\| P^{[m,n]} f - P^{[m,n]} g \right\|_1 = 0,$$

3. *strong asymptotically stable if there exists a unique $f_* \in \mathcal{D}$ such that for all $m \geq 0$ and $f \in \mathcal{D}$*

$$\lim_{n \rightarrow \infty} \left\| P^{[m,n]} f - f_* \right\|_1 = 0,$$

4. *strong almost asymptotically stable if for all $m \geq 0$ and $f, g \in \mathcal{D}$*

$$\lim_{n \rightarrow \infty} \left\| P^{[m,n]} f - P^{[m,n]} g \right\|_1 = 0.$$

The long-term behaviour of iterates of nonhomogeneous stochastic operators is still a subject of interest, despite having been intensively studied. Herkenrath (1988) provides a comprehensive review of different asymptotic behaviours of nonhomogeneous Markov chains. Very recently Mukhamedov (2013a,b,c) contributed further this direction.

One of the reasons for the growing interest in quadratic stochastic operators is their applicability to biological sciences, especially in the modelling of reproducing populations (Bartoszek and Pułka, 2013a; Ganikhodjaev et al., 2010, 2013, 2014; Ganikhodjaev and Jamilov, 2012; Rudnicki and Zwolenski, 2015; Zwolenski, 2015) and describing multi-agent systems (Saburov and Saburov, 2014a,b).

This interest has resulted in numerous recent works on general (Alexandrov, 2015; Mukhamedov and Supar, 2015; Mukhamedov et al., 2013) and ergodic (Ganikhodjaev and Zanin, 2012; Ganikhodjaev et al., 2006; Ganikhodzhaev and Zanin, 2004; Pułka, 2011; Saburov, 2007) properties of QSOs, specific subclasses of QSOs like Volterra (Mukhamedov, 2014; Mukhamedov and Saburov, 2010; Saburov, 2012, 2013) or other (Ganikhodjaev and Hamzah, 2014). These new results build on a rich previous literature (e.g. Ganikhodzhaev, 1989, 1993; Ganikhodzhaev and Eshmamatova, 2006; Sarymsakov and Ganikhodzhaev, 1990; Zakharevich, 1978). We now define a quadratic stochastic operator acting on the L^1 space (cf. Bartoszek and Pułka, 2015).

Definition 2.2 *A bilinear operator $\mathbf{Q}: L^1 \times L^1 \rightarrow L^1$ is called a quadratic stochastic operator if*

$$\mathbf{Q}(f, g) \geq 0, \quad \mathbf{Q}(f, g) = \mathbf{Q}(g, f) \quad \text{and} \quad \|\mathbf{Q}(f, g)\|_1 = \|f\|_1 \|g\|_1$$

for all $f, g \geq 0, f, g \in L^1$.

Notice that \mathbf{Q} is bounded as $\sup_{\|f\|_1=1, \|g\|_1=1} \|\mathbf{Q}(f, g)\|_1 = 1$. Moreover if $\tilde{f} \geq f \geq 0$ and $\tilde{g} \geq g \geq 0$ then $\mathbf{Q}(\tilde{f}, \tilde{g}) \geq \mathbf{Q}(f, g)$. Clearly, $\mathbf{Q}(\mathcal{D} \times \mathcal{D}) \subseteq \mathcal{D}$. The family of all quadratic stochastic operators will be denoted by \mathfrak{Q} .

In population genetics special attention is paid to a nonlinear mapping $\mathcal{D} \ni f \mapsto \mathbb{Q}(f) := \mathbf{Q}(f, f) \in \mathcal{D}$. The iterates $\mathbb{Q}^n(f)$, where $n = 0, 1, 2, \dots$, model the evolution of a distribution of some (continuous) trait of an inbreeding or hermaphroditic population. We (Bartoszek and Pułka, 2013a) previously discussed in detail biological interpretations of different types of asymptotic behaviours of quadratic stochastic operators in the discrete $\ell^1 = L^1(\mathbb{N}, 2^{\mathbb{N}}, \text{counting measure})$ case. Clearly, $\mathbb{Q}(\mathcal{D}) \subseteq \mathcal{D}$.

We (Bartoszek and Pułka, 2015) showed that \mathbf{Q} is continuous on $L^1 \times L^1$ and uniformly continuous if applied to vectors from the unit ball in L^1 . In particular, \mathbb{Q} is uniformly continuous on the unit ball in L^1 .

We will endow the set \mathfrak{Q} with a metric structure relevant to the uniform convergence on \mathcal{D} . Given $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathfrak{Q}$ let us define

$$(1) \quad d_u(\mathbf{Q}_1, \mathbf{Q}_2) = \sup_{f \in \mathcal{D}} \|\mathbb{Q}_1(f) - \mathbb{Q}_2(f)\|_1,$$

$$(2) \quad \hat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) = \sup_{f, g \in \mathcal{D}} \|\mathbf{Q}_1(f, g) - \mathbf{Q}_2(f, g)\|_1.$$

Notice that the above metrics are equivalent. Clearly we have

$$d_u(\mathbf{Q}_1, \mathbf{Q}_2) = \sup_{f \in \mathcal{D}} \|\mathbf{Q}_1(f) - \mathbf{Q}_2(f)\|_1 \leq \sup_{f, g \in \mathcal{D}} \|\mathbf{Q}_1(f, g) - \mathbf{Q}_2(f, g)\|_1 = \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2).$$

On the other hand, let

$$\|\mathbf{Q}_1(\tilde{f}, \tilde{g}) - \mathbf{Q}_2(\tilde{f}, \tilde{g})\|_1 \geq (1 - \varepsilon) \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2)$$

for some $\tilde{f}, \tilde{g} \in \mathcal{D}$ (where later on we let $\varepsilon \rightarrow 0^+$). If $\tilde{f} = \tilde{g} \mu$ - a.e. then

$$d_u(\mathbf{Q}_1, \mathbf{Q}_2) \geq \|\mathbf{Q}_1(\tilde{f}) - \mathbf{Q}_2(\tilde{f})\|_1 \geq \frac{1}{2} \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) \geq \frac{1}{4} \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2).$$

If $d_u(\mathbf{Q}_1, \mathbf{Q}_2) \leq \frac{1}{2} \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2)$ then by bilinearity of \mathbf{Q}_1 and \mathbf{Q}_2 for $h := \frac{1}{2}\tilde{f} + \frac{1}{2}\tilde{g} \in \mathcal{D}$ we have

$$\begin{aligned} d_u(\mathbf{Q}_1, \mathbf{Q}_2) &\geq \|\mathbf{Q}_1(h) - \mathbf{Q}_2(h)\|_1 \\ &= \left\| \mathbf{Q}_1\left(\frac{1}{2}\tilde{f} + \frac{1}{2}\tilde{g}\right) - \mathbf{Q}_2\left(\frac{1}{2}\tilde{f} + \frac{1}{2}\tilde{g}\right) \right\|_1 \\ &\geq \frac{1}{2} \left\| \mathbf{Q}_1(\tilde{f}, \tilde{g}) - \mathbf{Q}_2(\tilde{f}, \tilde{g}) \right\|_1 - \frac{1}{4} \left\| \mathbf{Q}_1(\tilde{f}) - \mathbf{Q}_2(\tilde{f}) \right\|_1 - \frac{1}{4} \left\| \mathbf{Q}_1(\tilde{g}) - \mathbf{Q}_2(\tilde{g}) \right\|_1 \\ &\geq \frac{1}{2} (1 - \varepsilon) \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) - \frac{1}{8} \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) - \frac{1}{8} \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) \\ &= \left(\frac{1}{2} - \frac{\varepsilon}{2} - \frac{1}{4} \right) \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2). \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ finally we obtain

$$\frac{1}{4} \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) \leq d_u(\mathbf{Q}_1, \mathbf{Q}_2) \leq \widehat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2).$$

A natural question arises concerning the necessity of the symmetry condition in the definition of a quadratic stochastic operator. Indeed, in general one could consider a *nonsymmetric quadratic stochastic operator*, i.e. a bilinear operator $\mathbf{Q}: L^1 \times L^1 \rightarrow L^1$ such that $\mathbf{Q}(f, g) \geq 0$ and $\|\mathbf{Q}(f, g)\|_1 = \|f\|_1 \|g\|_1$ for all $f, g \geq 0, f, g \in L^1$. However this situation is a topic for further research. A small number of fragments of Bartoszek and Pułka (2013b)'s and our (Bartoszek and Pułka,

2015)'s proofs do not follow through in this case. Furthermore we notice that d_u will not be a metric. This is as for $\mathbf{Q}^b, \mathbf{Q}^\sharp \in \mathfrak{Q}$ defined by $\mathbf{Q}^b(f, g) = f(\int_X g d\mu)$ and $\mathbf{Q}^\sharp(f, g) = g(\int_X f d\mu)$ for any $f, g \in L^1$ we have that both \mathbf{Q}^b and \mathbf{Q}^\sharp are nonsymmetric quadratic stochastic operators such that $\mathbf{Q}^b \neq \mathbf{Q}^\sharp$ and $d_u(\mathbf{Q}^b, \mathbf{Q}^\sharp) = \sup_{f \in \mathcal{D}} \|\mathbf{Q}^b(f) - \mathbf{Q}^\sharp(f)\|_1 = 0$. On the other hand, it is easy to check that \hat{d}_u is a metric in the class of nonsymmetric quadratic stochastic operators.

Different types of asymptotic behaviours of quadratic stochastic operators and the relationships between them have been recently intensively studied by Bartoszek and Pułka (2013b), Bartoszek and Pułka (2015) and Rudnicki and Zwoleński (2015). We follow Bartoszek and Pułka (2015) in defining a quadratic stochastic operator.

Definition 2.3 *A quadratic stochastic operator $\mathbf{Q} \in \mathfrak{Q}$ is called:*

1. *norm mixing (uniformly asymptotically stable) if there exists a density $f \in \mathcal{D}$ such that*

$$\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{D}} \|\mathbf{Q}^n(g) - f\|_1 = 0 ,$$

2. *strong mixing (asymptotically stable) if there exists a density $f \in \mathcal{D}$ such that for all $g \in \mathcal{D}$ we have*

$$\lim_{n \rightarrow \infty} \|\mathbf{Q}^n(g) - f\|_1 = 0 ,$$

3. *strong almost mixing if for all $g, h \in \mathcal{D}$ we have*

$$\lim_{n \rightarrow \infty} \|\mathbf{Q}^n(g) - \mathbf{Q}^n(h)\|_1 = 0 .$$

The sets of all norm mixing, strong mixing, strong almost mixing quadratic stochastic operators are denoted respectively by \mathfrak{Q}_{nm} , \mathfrak{Q}_{sm} , \mathfrak{Q}_{sam} . It can be easily seen (cf. Bartoszek and Pułka, 2015) that $\mathfrak{Q}_{nm} \subsetneq \mathfrak{Q}_{sm} \subsetneq \mathfrak{Q}_{sam}$.

We now introduce the relation between quadratic stochastic operators and (linear) Markov operators. Ganihodzhaev (1991) was the first to introduce this approach and it was used recently by Bartoszek and Pułka (2013b) and us (Bartoszek and Pułka, 2015). This correspondence allows one to study a linear model instead of a nonlinear one. Again we follow Bartoszek and Pułka (2015).

Definition 2.4 Let $\mathbf{Q} \in \mathfrak{Q}$. For arbitrarily fixed initial density function $g \in \mathcal{D}$ a nonhomogeneous Markov chain associated with \mathbf{Q} and $g \in \mathcal{D}$ is defined as a sequence $\mathbf{P}_g = (P_g^{[n,n+1]})_{n \geq 0}$ of Markov operators $P_g^{[n,n+1]}: L^1 \rightarrow L^1$ of the form

$$P_g^{[n,n+1]}(h) := \mathbf{Q}(\mathbb{Q}^n(g), h) .$$

Notice that if the initial density f is \mathbb{Q} -invariant (i.e. $\mathbb{Q}(f) = f$), then the associated Markov chain \mathbf{P}_f is homogeneous as for any $h \in L^1$ the expression $\mathbf{Q}(\mathbb{Q}^n(f), h) = \mathbf{Q}(f, h)$ does not depend on n . In this case we abbreviate the notation and write $P_f^{[n,n+1]} =: P_f$ and $P_f^{[0,n]} =: P_f^n$.

Norm mixing of a quadratic stochastic operator $\mathbf{Q} \in \mathfrak{Q}$ is evidently correlated with asymptotic behaviour of its associated nonhomogeneous Markov chain as we proved (Bartoszek and Pułka, 2015). Namely we have

Theorem 2.1 (Bartoszek and Pułka (2015)) *Let \mathbf{Q} be a quadratic stochastic operator. The following conditions are equivalent:*

(1) *There exists $f \in \mathcal{D}$ such that*

$$\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{D}} \|\mathbb{Q}^n(g) - f\|_1 = 0.$$

(2) *There exists $f \in \mathcal{D}$ such that*

$$\lim_{n \rightarrow \infty} \sup_{g, h \in \mathcal{D}} \|P_h^{[0,n]}(g) - f\|_1 = 0.$$

(3) *There exists $f \in \mathcal{D}$ such that for every $m \geq 0$ we have*

$$\lim_{n \rightarrow \infty} \sup_{g, h \in \mathcal{D}} \|P_h^{[m,n]}(g) - f\|_1 = 0,$$

i.e. independently of the seed $g \in \mathcal{D}$, all nonhomogeneous Markov chains $\mathbf{P}_g = (P_g^{[n,n+1]})_{n \geq 0}$ are norm mixing with a common limit distribution f and the rate of convergence is uniform for g .

3 Mutual correspondence between L^1 and ℓ^1 spaces and its consequences

Let us recall that a measurable countable partition $\xi := \{B_k\}_{k=1}^\infty$ of X is called consistent with σ -finite measure μ if $0 < \mu(B_k) < \infty$ for all k . Such partitions exist since the measure μ is σ -finite. Given a consistent measurable countable partition $\xi := \{B_k\}$ and any $f_1, f_2 \in L^1$ we write

$$f_1 \sim f_2 \Leftrightarrow \forall_k \int_{B_k} f_1 d\mu = \int_{B_k} f_2 d\mu$$

what defines an equivalence relationship on the L^1 space. Hence each equivalence class (taking $f \in L^1$ as its representative) can be associated with an element $p_f \in \ell^1$, namely take

$$\ell^1 \ni p_f = \left(\int_{B_1} f d\mu, \int_{B_2} f d\mu, \dots \right).$$

Notice that the coordinates of the vector p_f are actually the conditional expectations $E[\cdot | B_k]$ for the density f and measure μ .

Motivated by the mutual correspondence between L^1 and ℓ^1 we recall the definition of a quadratic stochastic operator on ℓ^1 (Bartoszek and Pułka, 2013b).

Definition 3.1 *A quadratic stochastic operator is defined as a cubic array of non-negative real numbers $\mathbf{Q}_{seq} = [q_{ij,k}]_{i,j,k \geq 1}$ if it satisfies*

$$(D1) \quad 0 \leq q_{ij,k} = q_{ji,k} \leq 1 \text{ for all } i, j, k \geq 1,$$

$$(D2) \quad \sum_{k=1}^\infty q_{ij,k} = 1 \text{ for any pair } (i, j).$$

Such a cubic matrix \mathbf{Q}_{seq} may be viewed as a bilinear mapping $\mathbf{Q}_{seq} : \ell^1 \times \ell^1 \rightarrow \ell^1$ if we set $\mathbf{Q}_{seq}((x_1, x_2, \dots), (y_1, y_2, \dots))_k = \sum_{i,j=1}^\infty x_i y_j q_{ij,k}$ for any $k \geq 1$.

We denote by $\mathfrak{Q} := \mathfrak{Q}(L^1)$ and $\mathfrak{Q}(\ell^1)$ the sets of all quadratic stochastic operators defined on L^1 and ℓ^1 respectively. Let $\mathbb{E} : \mathfrak{Q}(L^1) \rightarrow \mathfrak{Q}(\ell^1)$ be defined by

$$\mathbb{E}\mathbf{Q}(\underline{x}, \underline{y}) = \sum_{i=1}^\infty \left(\int_{B_i} \mathbf{Q} \left(\sum_{k=1}^\infty \frac{x_k}{\mu(B_k)} \mathbf{1}_{B_k}, \sum_{l=1}^\infty \frac{y_l}{\mu(B_l)} \mathbf{1}_{B_l} \right) d\mu \right) \underline{e}_i,$$

where $\underline{x} = (x_1, x_2, \dots), \underline{y} = (y_1, y_2, \dots) \in \ell^1$, $\underline{e}_i := (\delta_{ji})_{j \geq 1}$ and $\delta_{ji} = 1$ for $j = i$ and $\delta_{ji} = 0$ for $j \neq i$. Let us notice that the mapping \mathbb{E} is continuous. Indeed,

let $\mathbf{Q}_1 \in \mathfrak{Q}(L^1)$ and $\varepsilon > 0$ be fixed and choose $\delta = \varepsilon$. Then for any $\mathbf{Q}_2 \in \mathfrak{Q}(L^1)$ satisfying $d_u(\mathbf{Q}_1, \mathbf{Q}_2) = \sup_{f \in \mathcal{D}} \|\mathbf{Q}_1(f) - \mathbf{Q}_2(f)\|_1 < \delta$ we have

$$\begin{aligned}
\sup_{\{\underline{x} \in \ell^1 : \|\underline{x}\|_{\ell^1} = 1, x_k \geq 0\}} \|\mathbb{E}\mathbf{Q}_1(\underline{x}, \underline{x}) - \mathbb{E}\mathbf{Q}_2(\underline{x}, \underline{x})\|_{\ell^1} &\leq \sup_{f \in \mathcal{D}} \sum_{k=1}^{\infty} \left| \int_{B_k} \mathbf{Q}_1(f) d\mu - \int_{B_k} \mathbf{Q}_2(f) d\mu \right| \\
&\leq \sup_{f \in \mathcal{D}} \sum_{k=1}^{\infty} \int_{B_k} |\mathbf{Q}_1(f) - \mathbf{Q}_2(f)| d\mu \\
&= \sup_{f \in \mathcal{D}} \int_X |\mathbf{Q}_1(f) - \mathbf{Q}_2(f)| d\mu \\
&= \sup_{f \in \mathcal{D}} \|\mathbf{Q}_1(f) - \mathbf{Q}_2(f)\|_1 < \varepsilon.
\end{aligned}$$

Remark 3.1 *By the continuity of the mapping \mathbb{E} we obtain that if \mathcal{Q} is an open subset of $\mathfrak{Q}(\ell^1)$ then its preimage $\mathbb{E}^{-1}(\mathcal{Q})$ is an open subset in $\mathfrak{Q}(L^1)$. We will use this fact to describe the geometric structure of the set $\mathfrak{Q}(L^1)$.*

4 Prevalence problem in the set of quadratic stochastic operators

In this section we study whether there is any typical asymptotic behaviour of quadratic stochastic operators. We use symbols $\text{Int}(A)$, A^c and $\text{diam}(A)$ to denote the interior, the complement and the diameter of the set A . Of course the whole below discussion excludes the trivial cases of $\dim \ell^1 = 1$ and $\dim L^1 = 1$.

We begin our study with a description of the class of (uniformly) asymptotically stable quadratic stochastic operators. Recall that Bartoszek and Pułka (2013b) proved that the interior of the set $\mathfrak{Q}_{sm}^c(\ell^1)$, quadratic stochastic operators acting on ℓ^1 which are not strong mixing, is nonempty. Taking into account the mutual correspondence between the spaces L^1 and ℓ^1 discussed in the previous section as well as Remark 3.1 we obtain the following

Corollary 4.1 *The set $\mathbb{E}^{-1}(\text{Int}(\mathfrak{Q}_{sm}^c(\ell^1)))$ is nonempty, open and moreover*

$$\mathbb{E}^{-1}(\text{Int}(\mathfrak{Q}_{sm}^c(\ell^1))) \subseteq \mathfrak{Q}_{sm}^c.$$

In particular, \mathfrak{Q}_{nm}^c contains a nonempty open set.

On the other hand, Bartoszek and Pułka (2013b) also showed that the interior of the set $\mathfrak{Q}_{nm}^c(\ell^1)$, quadratic stochastic operators acting on ℓ^1 which are norm

mixing, is nonempty as well. We will show directly that $\text{Int}(\mathfrak{Q}_{nm}(L^1))$ is non-empty.

Example 4.1 Let us define $\mathbf{Q}^\diamond \in \mathfrak{Q}$ by $\mathbf{Q}^\diamond(f, g) = (\int_X f d\mu)(\int_X g d\mu)h$, for any $f, g \in L^1$ and a fixed density function $h \in \mathcal{D}$. Clearly \mathbf{Q}^\diamond is norm mixing. Suppose that $\mathbf{Q} \in \mathfrak{Q}$ satisfies $\widehat{d}_u(\mathbf{Q}, \mathbf{Q}^\diamond) = \varepsilon < \frac{1}{3}$. We will show that such a \mathbf{Q} is also norm mixing. For any $f, g, u, v \in \mathcal{D}$ we have

$$\|\mathbf{Q}(f, g) - \mathbf{Q}(u, v)\|_1 \leq \|\mathbf{Q}(f, g) - \mathbf{Q}^\diamond(f, g)\|_1 + \|\mathbf{Q}(u, v) - \mathbf{Q}^\diamond(u, v)\|_1 \leq 2\varepsilon.$$

Thus, $\text{diam}(\overline{\mathbb{Q}(\mathcal{D})}) \leq 2\varepsilon$. Let $u, v \in \overline{\mathbb{Q}(\mathcal{D})}$ and so $\|u - v\|_1 = \kappa \leq 2\varepsilon$. We denote by \wedge the ordinary minimum in L^1 and let $f = u - u \wedge v$, $g = v - u \wedge v$. Since L^1 is an AL-space, then $\|u \wedge v\|_1 = 1 - \frac{\kappa}{2}$ and $\|g\|_1 = \|f\|_1 = \frac{\kappa}{2}$. We have

$$\begin{aligned} \|\mathbb{Q}(u) - \mathbb{Q}(v)\|_1 &= \|\mathbb{Q}(u \wedge v + f) - \mathbb{Q}(u \wedge v + g)\|_1 \\ &= \|2\mathbf{Q}(u \wedge v, f) + \mathbb{Q}(u \wedge v) + \mathbb{Q}(f) - 2\mathbf{Q}(u \wedge v, g) - \mathbb{Q}(u \wedge v) - \mathbb{Q}(g)\|_1 \\ &\leq 2\|\mathbf{Q}(u \wedge v, f) - \mathbf{Q}(u \wedge v, g)\|_1 + \|\mathbb{Q}(f)\|_1 + \|\mathbb{Q}(g)\|_1 \\ &= 2\frac{\kappa}{2}\left(1 - \frac{\kappa}{2}\right) \left\| \mathbf{Q}\left(\frac{u \wedge v}{1 - \kappa/2}, \frac{f}{\kappa/2}\right) - \mathbf{Q}\left(\frac{u \wedge v}{1 - \kappa/2}, \frac{g}{\kappa/2}\right) \right\|_1 + \frac{\kappa^2}{2} \\ &\leq \kappa\left(1 - \frac{\kappa}{2}\right) 2\varepsilon + \frac{\kappa^2}{2} < 2\kappa\varepsilon + \frac{\kappa^2}{2} \\ &= \left(2\varepsilon + \frac{\kappa}{2}\right) \kappa = \left(2\varepsilon + \frac{\kappa}{2}\right) \|u - v\|_1 \leq (2\varepsilon + \varepsilon) \|u - v\|_1 = 3\varepsilon \|u - v\|_1. \end{aligned}$$

It follows that \mathbb{Q} is a strict contraction. Applying Banach's fixed point theorem we obtain

$$\sup_{u, v \in \mathcal{D}} \|\mathbb{Q}^n(u) - \mathbb{Q}^n(v)\|_1 \leq 2(3\varepsilon)^{n-1} \rightarrow 0,$$

which gives that \mathbf{Q} is also norm mixing. Hence, $\text{Ball}(\mathbf{Q}^\diamond, \frac{1}{3}) \subseteq \text{Int}(\mathfrak{Q}_{nm})$.

Corollary 4.2 $\text{Int}(\mathfrak{Q}_{nm}) \neq \emptyset$.

Hence neither norm mixing nor non norm mixing quadratic stochastic operators can be considered as generic. We introduce another type of asymptotic behaviour of quadratic stochastic operators and show that it is prevalent in \mathfrak{Q} .

Definition 4.1 We say that $\mathbf{Q} \in \mathfrak{Q}$ is norm quasi-mixing if

$$\lim_{n \rightarrow \infty} \sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[0, n]}(g) - P_f^{[0, n]}(h) \right\|_1 = 0.$$

The set of all norm quasi-mixing quadratic stochastic operators will be denoted by \mathfrak{Q}_{nqm} .

Let us note that the norm quasi-mixing condition is equivalent to

$$\lim_{n \rightarrow \infty} \sup_{f, g \in \mathcal{D}} \left\| P_f^{[0, n]}(g) - \mathbb{Q}^n(f) \right\|_1 = 0.$$

Indeed, to see the necessity of the above condition it is enough to substitute $h = f$ as $P_f^{[0, n]}(f) = \mathbb{Q}^n(f)$. The sufficiency follows directly from the triangle inequality, namely

$$\sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[0, n]}(g) - P_f^{[0, n]}(h) \right\|_1 \leq \sup_{f, g \in \mathcal{D}} \left\| P_f^{[0, n]}(g) - \mathbb{Q}^n(f) \right\|_1 + \sup_{f, h \in \mathcal{D}} \left\| P_f^{[0, n]}(h) - \mathbb{Q}^n(f) \right\|_1.$$

Clearly norm mixing quadratic stochastic operators are also norm quasi-mixing, since according to the Theorem 2.1 we have

$$\sup_{g, h \in \mathcal{D}} \left\| P_h^{[0, n]}(g) - \mathbb{Q}^n(h) \right\|_1 \leq \sup_{g, h \in \mathcal{D}} \left\| P_h^{[0, n]}(g) - f \right\|_1 + \sup_{h \in \mathcal{D}} \left\| \mathbb{Q}^n(h) - f \right\|_1 \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand, norm-quasi mixing does not imply strong (and hence norm) mixing. This can be seen in the example below.

Example 4.2 Given a (homogeneous) Markov operator $P: L^1 \rightarrow L^1$ let us define $\mathbf{Q} \in \mathfrak{Q}$ by

$$\mathbf{Q}(f, g) = \frac{1}{2} \left(\left(\int_X g d\mu \right) Pf + \left(\int_X f d\mu \right) Pg \right)$$

for any $f, g \in L^1$. Then for a fixed density $f \in \mathcal{D}$ and any $g \in \mathcal{D}$ we have $P_f(g) = \mathbf{Q}(f, g) = \frac{1}{2}(Pf + Pg)$. Thus using the fact that P is a strict contraction we get

$$\left\| P_f(g) - P_f(h) \right\|_1 = \frac{1}{2} \left\| Pg - Ph \right\|_1 \leq \frac{1}{2} \left\| g - h \right\|_1.$$

Similarly, for any natural n we have

$$\begin{aligned}\left\|P_f^{[0,n]}(g) - P_f^{[0,n]}(h)\right\|_1 &= \left\|\frac{1}{2} \left(P(P_f^{[0,n-1]}(g)) - P(P_f^{[0,n-1]}(h))\right)\right\|_1 \\ &\leq \frac{1}{2} \left\|P_f^{[0,n-1]}(g) - P_f^{[0,n-1]}(h)\right\|_1 \leq \dots \leq \frac{1}{2^n} \|f - h\|_1 .\end{aligned}$$

Thus

$$\sup_{f,g,h \in \mathcal{D}} \left\|P_f^{[0,n]}(g) - P_f^{[0,n]}(h)\right\|_1 \leq \frac{1}{2^{n-1}} \xrightarrow{n \rightarrow \infty} 0$$

and hence \mathbf{Q} is norm quasi-mixing. On the other hand, let us assume that the Markov operator P does not possess any invariant density. Since $\mathbb{Q}(f) = Pf$ for any $f \in \mathcal{D}$, then neither \mathbb{Q} has invariant densities. In particular \mathbf{Q} is not strong (and hence not norm) mixing.

Below we present our main result. We show that \mathfrak{Q}_{nqm} is a large set in the topology induced by metric d_u and hence norm quasi-mixing can be considered as a generic property in the class of quadratic stochastic operators.

Theorem 4.1 \mathfrak{Q}_{nqm} is a dense and open subset for the metric d_u .

PROOF We first show that \mathfrak{Q}_{nqm} is a dense subset of \mathfrak{Q} for the metric d_u . Let $\mathbf{Q} \in \mathfrak{Q}$ be taken arbitrarily. For any $f, g \in L^1$ and any fixed density $v \in \mathcal{D}$ define, similarly as in Ex. 4.1, $\mathbf{Q}^\diamond \in \mathfrak{Q}$ by

$$\mathbf{Q}^\diamond(f, g) := \left(\int_X f d\mu \int_X g d\mu \right) v .$$

For any $0 < \varepsilon < 1$ define $\mathbf{Q}_\varepsilon \in \mathfrak{Q}$ by

$$\mathbf{Q}_\varepsilon = (1 - \varepsilon)\mathbf{Q} + \varepsilon\mathbf{Q}^\diamond .$$

Then

$$\begin{aligned}d_u(\mathbf{Q}, \mathbf{Q}_\varepsilon) \leq \hat{d}_u(\mathbf{Q}, \mathbf{Q}_\varepsilon) &= \sup_{f,g \in \mathcal{D}} \|\mathbf{Q}(f, g) - \mathbf{Q}_\varepsilon(f, g)\|_1 \\ &= \sup_{f,g \in \mathcal{D}} \|\mathbf{Q}(f, g) - (1 - \varepsilon)\mathbf{Q}(f, g) - \varepsilon\mathbf{Q}^\diamond(f, g)\|_1 \leq 2\varepsilon .\end{aligned}$$

Moreover, \mathbf{Q}_ε is norm quasi-mixing. Indeed, consider nonhomogeneous Markov chain ${}_\varepsilon \mathbf{P}_f = ({}_\varepsilon P_f^{[n,n+1]})_{n \geq 0}$ associated with \mathbf{Q}_ε and $f \in \mathcal{D}$. For any $g, h \in \mathcal{D}$ we have

$$\begin{aligned}
& \left\| {}_\varepsilon P_f^{[0,n]}(g) - {}_\varepsilon P_f^{[0,n]}(h) \right\|_1 \\
&= \left\| \mathbf{Q}_\varepsilon(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(g)) - \mathbf{Q}_\varepsilon(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(h)) \right\|_1 \\
&= \left\| (1-\varepsilon)\mathbf{Q}(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(g)) + \varepsilon v - (1-\varepsilon)\mathbf{Q}(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(h)) - \varepsilon v \right\|_1 \\
&= (1-\varepsilon) \left\| \mathbf{Q}(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(g)) - \mathbf{Q}(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(h)) \right\|_1 \\
&= (1-\varepsilon) \left\| \mathbf{Q}(\mathbb{Q}_\varepsilon^{n-1}(f), {}_\varepsilon P_f^{[0,n-1]}(g) - {}_\varepsilon P_f^{[0,n-1]}(h)) \right\|_1 \\
&\leq (1-\varepsilon) \left\| {}_\varepsilon P_f^{[0,n-1]}(g) - {}_\varepsilon P_f^{[0,n-1]}(h) \right\|_1 \leq \dots \leq 2(1-\varepsilon)^n
\end{aligned}$$

and hence

$$\sup_{g, h \in \mathcal{D}} \left\| {}_\varepsilon P_f^{[0,n]}(g) - {}_\varepsilon P_f^{[0,n]}(h) \right\|_1 \leq 2(1-\varepsilon)^n \xrightarrow{n \rightarrow \infty} 0.$$

Therefore \mathfrak{Q}_{nqm} is dense in \mathfrak{Q} .

We now show that \mathfrak{Q}_{nqm} is an open set for the metric d_u . We first notice that for a nonhomogeneous Markov chain $\mathbf{P}_f = (P_f^{[n,n+1]})_{n \geq 0}$ associated with $\mathbf{Q} \in \mathfrak{Q}$ and $f \in \mathcal{D}$ the sequence

$$\sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1$$

is non-increasing in n as

$$\begin{aligned}
\sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[0,n+1]}(g) - P_f^{[0,n+1]}(h) \right\|_1 &= \sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[n,n+1]}(P_f^{[0,n]}(g) - P_f^{[0,n]}(h)) \right\|_1 \\
&\leq \sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1.
\end{aligned}$$

Hence $\mathbf{Q} \in \mathfrak{Q}$ is norm quasi-mixing if and only if the limit condition is satisfied on some subsequence. We will show that

$$\mathfrak{Q}_{nqm} = \bigcup_{n=1}^{\infty} \left\{ \mathbf{Q} \in \mathfrak{Q} : \sup_{f, g, h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1 < 2 \right\}.$$

The inclusion

$$\mathfrak{Q}_{nqm} \subseteq \bigcup_{n=1}^{\infty} \left\{ \mathbf{Q} \in \mathfrak{Q} : \sup_{f,g,h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1 < 2 \right\}$$

is obvious. In order to prove the opposite inclusion let us assume that there exists a natural n such that $\mathbf{Q} \in \mathfrak{Q}$ satisfies

$$\sup_{f,g,h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1 < 2 - \varepsilon$$

for arbitrarily small $\varepsilon > 0$.

Using the fact that L^1 is an AL-space, the above inequality can be written in an equivalent form

$$\inf_{f,g,h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) \wedge P_f^{[0,n]}(h) \right\|_1 > \frac{\varepsilon}{2}.$$

We introduce the following notation (for $P_f^{[0,k]}(g) \neq P_f^{[0,k]}(h)$)

$$\begin{aligned} \mathfrak{J}(k, f, g, h) &:= \frac{P_f^{[0,k]}(g) - P_f^{[0,k]}(g) \wedge P_f^{[0,k]}(h)}{1 - \left\| P_f^{[0,k]}(g) \wedge P_f^{[0,k]}(h) \right\|_1} - \frac{P_f^{[0,k]}(h) - P_f^{[0,k]}(g) \wedge P_f^{[0,k]}(h)}{1 - \left\| P_f^{[0,k]}(g) \wedge P_f^{[0,k]}(h) \right\|_1} \\ &= \frac{P_f^{[0,k]}(g) - P_f^{[0,k]}(h)}{1 - \left\| P_f^{[0,k]}(g) \wedge P_f^{[0,k]}(h) \right\|_1} \in \mathcal{D} - \mathcal{D}. \end{aligned}$$

For any natural j we have

$$\begin{aligned} \left\| P_f^{[0,jn]}(g) - P_f^{[0,jn]}(h) \right\|_1 &= \left\| P_{\mathbb{Q}^n(f)}^{[0,n(j-1)]}(P_f^{[0,n]}(g) - P_f^{[0,n]}(h)) \right\|_1 \\ &= \left\| P_{\mathbb{Q}^n(f)}^{[0,n(j-1)]} \left(1 - \left\| P_f^{[0,n]}(g) \wedge P_f^{[0,n]}(h) \right\|_1 \right) \mathfrak{J}(n, f, g, h) \right\|_1 \\ &\leq \left(1 - \frac{\varepsilon}{2} \right) \left\| P_{\mathbb{Q}^n(f)}^{[0,n(j-1)]}(\mathfrak{J}(n, f, g, h)) \right\|_1 \leq \dots \leq 2 \left(1 - \frac{\varepsilon}{2} \right)^{j-1}. \end{aligned}$$

Thus

$$\sup_{f,g,h \in \mathcal{D}} \left\| P_f^{[0,jn]}(g) - P_f^{[0,jn]}(h) \right\|_1 \leq 2 \left(1 - \frac{\varepsilon}{2} \right)^{j-1} \xrightarrow{j \rightarrow \infty} 0$$

and hence $\mathbf{Q} \in \mathfrak{Q}_{nqm}$. Now let us notice that for any fixed natural n the function

$$\mathfrak{Q} \ni \mathbf{Q} \mapsto \sup_{f,g,h \in \mathcal{D}} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1$$

is continuous in the metric d_u . Indeed, let $f, g \in \mathcal{D}$ be fixed. Consider the family of functions

$$\mathcal{F}_{f,g} := \{ F_{f,g} : (\mathfrak{Q}, d_u) \ni \mathbf{Q} \mapsto \mathbf{Q}(f, g) \in (L^1, \|\cdot\|_1) \} .$$

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{4}$. For any $\mathbf{Q}_1 \in \mathfrak{Q}$ and $F_{f,g} \in \mathcal{F}_{f,g}$, the inequality $d_u(\mathbf{Q}_1, \mathbf{Q}_2) < \delta$ implies that

$$\begin{aligned} \|F_{f,g}(\mathbf{Q}_1) - F_{f,g}(\mathbf{Q}_2)\|_1 &= \|\mathbf{Q}_1(f, g) - \mathbf{Q}_2(f, g)\|_1 \\ &\leq \hat{d}_u(\mathbf{Q}_1, \mathbf{Q}_2) \leq 4d_u(\mathbf{Q}_1, \mathbf{Q}_2) < \varepsilon . \end{aligned}$$

Hence $\mathcal{F}_{f,g}$ is equicontinuous. Since

$$\begin{aligned} &| \|\mathbf{Q}_1(f, g) - \mathbf{Q}_1(f, h)\|_1 - \|\mathbf{Q}_2(f, g) - \mathbf{Q}_2(f, h)\|_1 | \\ &\leq \|\mathbf{Q}_1(f, g) - \mathbf{Q}_2(f, g)\|_1 + \|\mathbf{Q}_1(f, h) - \mathbf{Q}_2(f, h)\|_1 \end{aligned}$$

for any $f, g, h \in \mathcal{D}$ and $\text{diam}(\mathcal{D}) < \infty$ then the function

$$\mathfrak{Q} \ni \mathbf{Q} \mapsto \sup_{f,g,h \in \mathcal{D}} \|\mathbf{Q}(f, g) - \mathbf{Q}(f, h)\|_1 = \sup_{f,g,h \in \mathcal{D}} \|F_{f,g}(\mathbf{Q}) - F_{f,h}(\mathbf{Q})\|_1$$

is well defined and continuous in the metric d_u . In particular, the function

$$\mathfrak{Q} \ni \mathbf{Q} \mapsto \sup_{f,g,h \in \mathcal{D}} \left\| \mathbf{Q}(\mathbb{Q}^{n-1}(f), P_f^{[0,n-1]}(g)) - \mathbf{Q}(\mathbb{Q}^{n-1}(f), P_f^{[0,n-1]}(h)) \right\|_1$$

is continuous in the metric d_u . Thus we showed that for any fixed natural n the set

$$\left\{ \mathbf{Q} \in \mathfrak{Q} : \sup_{f,g,h} \left\| P_f^{[0,n]}(g) - P_f^{[0,n]}(h) \right\|_1 < 2 \right\}$$

is an open subset of \mathfrak{Q} in the metric d_u . □

It is worth emphasizing that even though \mathfrak{Q}_{nqm} is a large set, not every quadratic stochastic operator is norm quasi-mixing. This is seen in the following example.

Example 4.3 Let us consider a partition $X := B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$. Define the operator $\mathbf{Q} \in \mathfrak{Q}$ for any $f, g \in L^1$ and some fixed $h \in L^1$, $h \geq 0$, by

$$\begin{aligned} \mathbf{Q}(f, g) &:= \frac{h\mathbf{1}_{B_1}}{\int_{B_1} h d\mu} \left(\int_{B_1} f d\mu \int_{B_1} g d\mu + \int_{B_1} f d\mu \int_{B_2} g d\mu + \int_{B_2} f d\mu \int_{B_1} g d\mu \right) \\ &+ \frac{h\mathbf{1}_{B_2}}{\int_{B_2} h d\mu} \int_{B_2} f d\mu \int_{B_2} g d\mu. \end{aligned}$$

Denote by $\text{supp}(f) := \{x \in X : f(x) \neq 0 \text{ } \mu \text{ a.e.}\}$ the support of $f \in L^1$. For $f \in \mathcal{D}$ such that $\text{supp}(f) \subseteq B_2$ we have

$$\mathbf{Q}(f, g) = \frac{h\mathbf{1}_{B_1}}{\int_{B_1} h d\mu} \int_{B_1} g d\mu + \frac{h\mathbf{1}_{B_2}}{\int_{B_2} h d\mu} \int_{B_2} g d\mu.$$

Thus if $f, g \in L^1$ satisfy $\text{supp}(f) \subseteq B_2$ and $\text{supp}(g) \subseteq B_1$ then $P_f^{[0,n]}(f) = \frac{h\mathbf{1}_{B_2}}{\int_{B_2} h d\mu}$ and $P_f^{[0,n]}(g) = \frac{h\mathbf{1}_{B_1}}{\int_{B_1} h d\mu}$. Hence

$$\left\| P_f^{[0,n]}(f) - P_f^{[0,n]}(g) \right\|_1 = 2.$$

It follows that \mathbf{Q} is not norm quasi-mixing.

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